Analysis of structural reliability under parameter uncertainties

Armen Der Kiureghian

Department of Civil and Environmental Engineering, University of California, 723 Davis Hall, Berkeley, CA 94116, United States

Received 15 July 2007; accepted 19 October 2007
Available online 4 March 2008

Abstract

Formulation of structural reliability requires selection of probabilistic or physical models, which usually involve parameters to be estimated through statistical inference — a process that invariably introduces uncertainties in parameter estimates. The measure of reliability that incorporates these parameter uncertainties is termed the predictive reliability index. Methods for computing this measure and the corresponding failure probability are introduced. A simple approximate formula is derived for the predictive reliability index, which requires a single solution of the reliability problem together with parameter sensitivities with respect to mean parameter values. The approach also provides measures of the uncertainties inherent in the estimates of the reliability index and the failure probability, which arise from parameter uncertainties. An illustrative example involving component and system problems demonstrates the influence of parameter uncertainties on the predictive reliability index and the accuracy of the simple approximation formula.

1. Prologue

It is a privilege to be contributing to this special issue of Probabilistic Engineering Mechanics honoring the distinguished career and seminal contributions of Ove Dalager Ditlevsen. My first impressions of Ove are from September 1982, when I heard him give a series of lectures on structural reliability at a NATO Advanced Study Institute held in Bornholm, Denmark. Over the years, reading his papers and having discussions with him have profoundly influenced my own thinking about probability theory, the nature of uncertainties and the principles and methods of structural reliability analysis. For that reason, I wish to start this paper by briefly recounting from a series of communications with him on the topic of uncertainties and structural reliability during the summer of 1988. Provoked by a report I had written [4], which was later published in the ASCE Journal of Engineering Mechanics [5], the correspondence consisted of 12 letters, a total of 48 partly handwritten pages and 6 attachments, including detailed derivations and calculations (see Fig. 1). These communications are directly relevant to the topic of this paper.

E-mail address: adk@ce.Berkeley.edu.

Fig. 1. Correspondence with Ove Ditlevsen in summer of 1988.

In [4,5], I had investigated several measures of structural safety relative to a set of postulated “fundamental requirements” and advocated one measure, which included a penalty for uncertainty in the estimate of the reliability index arising from statistical and model uncertainties. The argument was based on a crisp distinction of aleatory and epistemic uncertainties — inherent variabilities (randomness) being aleatory in nature and model and statistical uncertainties being epistemic in
nature. It was argued that, since epistemic uncertainties could be reduced, the penalty would provide an incentive to gather additional data or use more refined models, thus satisfying a requirement of "remunerability", which stated "The reliability index shall remunerate improvements in the state of knowledge". The main points of the discussion in the communications with Ove were the nature of uncertainties and the measure of reliability to be used in structural design.

In his first letter in the series dated May 16, 1988, Ove started by "I have studied your report... with interest, and I have some comments on points where I disagree with you." With his characteristic modesty, he added "In the following, please put in the words 'I think that' before all statements formulated in absolute terms". His first disagreement was on the nature of inherent variabilities. He wrote: "Inherent variability is relative to a level of refinement of the model. Except for quantum mechanical phenomena inherent variability is reducible by more detailed modeling combined with corresponding gathering of information. For example, this is the case for material properties like stress–strain relationships. Thus, the inherent variability of material properties in structural reliability is relative to the 'usual' level of modeling material behavior. The same applies to the loads. For example, the inherent variability of the wind load on a structure can in principle be reduced as the abilities of meteorologists to predict wind velocities from geophysical models become better and better. Philosophically it does not make sense to claim that wind velocities are random. Randomness is not a property of nature but a property of the model".

Having questioned the crisp distinction between aleatory and epistemic uncertainties, Ove went on to say: "On a pragmatic basis I agree that it is useful to distinguish between inherent variability and model uncertainty, but this is relative to a given framework of engineering modeling practice... By distinguishing the intrinsic variability from uncertainty due to estimation error and model imperfection, it is on a pragmatic level clarified [as to] what can be done within a framework of usual practice by information gathering in order to (hopefully) increase safety without changing the structure." The parenthetical adverb "hopefully" referred to the fact that the additional information might in fact indicate that the structure is less safe than estimated based on the prior information. He explained "by bad luck the sample average may change with sample size so that it can happen that the reliability index decreases. This event will generally be more rare than the event of having an increase of the reliability index".

In the report and paper, I had also defined a requirement of "completeness" as "The reliability index shall incorporate all available information and account for all uncertainties". Ove wrote: "To talk about all uncertainties makes only sense relative to some model in which the uncertainties are represented by suitable formal elements. Otherwise a requirement containing the word 'all' becomes a religious commandment".

The fundamental point Ove was raising is that the nature of uncertainties can only be discussed within the confines of a model. Since our correspondence in 1988, I have come to fully appreciate this point. As we described in a recent co-authored paper [6] "Engineering problems, including reliability, risk and decision problems, without exception, are solved within the confines of a model universe. This universe contains the set of physical and probabilistic models, which are employed as mathematical idealizations of reality to render a solution for the problem at hand. The model universe may contain inherently uncertain quantities; furthermore, the sub-models are invariably imperfect giving rise to additional uncertainties. Therefore, an important part of building the model universe is the modeling of these uncertainties. Any discussion on the nature and character of uncertainties should be stated within the confines of the model universe".

Ove's second point of disagreement with my paper was my proposed "minimum-penalty reliability index". He did not object to the six fundamental requirements I had postulated for the measure of safety, except to suggest rewording of the completeness requirement as mentioned above. However, he insisted that what I had called "predictive reliability index" satisfied all the requirements and that it was the appropriate one to use within the context of decision making for structural design or code development. He also pointed out that this measure was identical to what he had earlier called "generalized reliability index" with statistical uncertainty taken into account [7–9]. I have come to agree with this point of view as well. In fact, one of the objectives of this paper is to formally define this measure of reliability and suggest several methods for its computation.

The importance of Ove Ditlevsen's work is not limited to his enormous collection of published papers and books, many of which contain seminal contributions to the theory of structural reliability, stochastic mechanics or the principles behind development of probabilistic codes. These are readily quantifiable. What is less easily quantifiable is his role as a corrective agent in the course of development of these fields. He has been a vocal critic of work by his colleagues (we have all seen him in this role in conferences and workshops), but his criticism has always been honest and constructive. In a sense, he has shepherded the advancement of the field on a healthy path to its present maturity. For this and many other reasons, it is a true privilege to contribute to this issue honoring his career.

2. Objectives and scope

The focus of this paper is on the predictive reliability index, or the generalized reliability index in Ove Ditlevsen's terminology, which is a measure of structural reliability that incorporates the influence of parameter uncertainties. The parameters are those inherent in the probabilistic or physical models employed in the formulation of the reliability model. After a brief review of the formulation of structural reliability problems under parameter uncertainties, the predictive failure probability and reliability index are formally defined and several methods for their computation are described. One method leads to a simple approximation of the predictive reliability index, which involves little more than a single reliability and sensitivity analysis using mean values of the model parameters. An additional advantage of this approach
is that it also provides a measure of the uncertainty in the reliability index, which arises from parameter uncertainties. An illustrative example demonstrates the influence of parameter uncertainties on the predictive reliability index and the accuracy of the simple approximation formula.

3. Formulation of structural reliability

The theoretical, time-invariant structural reliability problem is defined by the integral

\[ p = \int_{\Omega(x)} f_X(x) dx \]  

where \( p \) is the failure probability, \( f_X(x) \) is the joint probability density function (PDF) of a vector of random variables \( X = [X_1, X_2, \ldots, X_n]^T \) representing uncertain quantities such as loads, material property constants and geometric dimensions, and \( \Omega(x) \) is the failure domain of the structure in the outcome space \( x = [x_1, x_2, \ldots, x_n]^T \) of \( X \). For a general structural system, the failure domain may be defined in the form

\[ \Omega(x) = \bigcup_k \bigcap_i \{ g_i(x) \leq 0 \} \]  

where \( g_i(x) \), \( i = 1, \ldots, m \), are a set of limit-state functions formulated so that \( g_i(x) \leq 0 \) defines the failure of component \( i \), \( m \) denotes the number of components, and \( C_k \) is the index set for the \( k \)-th minimum cut set, where each minimum cut set represents a minimal set of components whose joint failure constitutes failure of the structural system. In the special case where each component is a cut set (failure of any component constitutes failure of the system), the structure is a series system. Conversely, when all components collectively form a single minimum cut set, the structure constitutes a parallel system.

We have termed the above a theoretical formulation of the structural reliability problem. This is because in practice neither the joint PDF \( f_X(x) \) nor the limit-state functions \( g_i(x) \) are precisely known. Instead, idealized models of these functions must be developed, typically through statistical inference using real-world observations complemented by engineering judgment. Let \( f_X(x|\Theta_f) \) denote the selected model for \( f_X(x) \), in which \( \Theta_f \) denotes a set of parameters to be estimated from observational data. Each of the limit-state functions \( g_i(x) \) is typically composed of several sub-models, e.g., separate capacity and demand models. However, for the sake of simplicity of the notation, here we assume each limit-state function is a single model. Thus, let \( \hat{g}_i(x, \Theta_{gi}) + E_i \) be the idealized model for \( g_i(x) \), in which \( \hat{g}_i(x, \Theta_{gi}) \) is a parameterized deterministic model with \( \Theta_{gi} \) denoting its parameters and \( E_i = g_i(x) - \hat{g}_i(x, \Theta_{gi}) \) is the residual, i.e., the error in the model for given \( x \) and \( \Theta_{gi} \). The model parameters \( \Theta_{gi} \), as well as the statistics of the model error \( E_i \), normally are estimated by statistical inference of laboratory or field data. It is usually reasonable to assume, possibly after an appropriate transformation of the model and the data, that the residual has the normal distribution with a standard deviation \( D_i \), which is independent of \( x \). Furthermore, with the aim of developing an unbiased model, the mean of \( E_i \) is set equal to zero. When multiple models are to be developed, it is additionally necessary to estimate the cross-correlations \( \rho_{ij}, i \neq j \), between pairs of model residuals. The collection of standard deviations \( D_i \) and correlation coefficients \( \rho_{ij} \) then forms the covariance matrix \( \Sigma \), which completely defines the joint distribution of the vector of model residuals \( E = [E_1, E_2, \ldots, E_m]^T \). The set of random variables of the reliability problem now is \( \{X, E\} \). Furthermore, the problem involves the set \( \Theta = (\Theta_f, \Theta_g, \Sigma) \) of model parameters, where \( \Theta_g \) is the collection of \( \Theta_{gi} \) for all limit-state models. As just mentioned, these parameters need to be estimated from observational data – a process which invariably introduces additional uncertainties. With increasing observational data, one can reduce uncertainties in parameter estimates; however, uncertainties in \( X \) and \( E \) cannot be influenced without changing the selected models. The main focus of this paper is on the influence of parameter uncertainties on the reliability estimate.

Consistent with the Bayesian notion of probability, unknown model parameters \( \Theta \) are considered as random variables. Their distribution is determined through the well known Bayesian updating formula

\[ f_{\Theta}(\theta|z) \propto L(z|\theta)f_{\Theta}(\theta) \]  

where \( f_{\Theta}(\theta) \) is the prior distribution and represents information available on parameters before observations are made, \( L(z|\theta) \) is the likelihood function for the set of observations \( z \) (a function proportional to the probability of making the observations \( z \) given \( \Theta = \theta \)), and \( f_{\Theta}(\theta|z) \) is the posterior distribution reflecting the updated information on \( \Theta \) in light of observations \( z \). This formula can be used to estimate distribution parameters \( \Theta_f \) as well as physical model parameters \( \Theta_g, \Sigma \). In general estimates for the two sets are statistically independent; however, statistical dependence may exist among elements of \( \Theta_g \) and \( \Sigma \). Examples for the estimation of distribution parameters can be found in many textbooks [1,2]. Specific examples in structural reliability for the estimation of physical model parameters or limit-state function parameters can be found in [10] and [13], respectively. In the following discussion, we drop the reference to the observations \( z \) and use the expression \( f_m(\theta) \) as the (updated) posterior PDF of the parameters \( \Theta \).

Using the parameterized distribution model of \( X \) in (1) and including random variables \( E \), the expression for failure probability becomes

\[ p(\Theta) = \int_{\Omega(x)} \int_{\varepsilon(x, \Theta_g, \Sigma)} \tilde{f}_X(x|\Theta_f)f_E(\varepsilon|\Sigma) dx d\varepsilon \]  

where \( \tilde{f}_X(x|\Theta_f) \) is the joint normal PDF of \( E \) conditioned on the covariance matrix \( \Sigma \) and

\[ \tilde{\Omega}(x, \varepsilon, \Theta_g) = \bigcup_k \bigcap_i \{ \hat{g}_i(x, \Theta_{gi}) + \varepsilon_i \leq 0 \} \]  

is the parameterized failure domain in the space of expanded random variables \( \{X, E\} \). It is seen that the failure probability is a function of the model parameters \( \Theta = (\Theta_f, \Theta_g, \Sigma) \). It follows that, since the model parameters are uncertain, the failure probability estimate is also uncertain. Consistent with the
Bayesian notion, we introduce $P = p(\Theta)$ as a random variable representing the uncertain failure probability. We also introduce the corresponding uncertain reliability index $B = \beta(\Theta)$, where $\beta(\Theta) = \Phi^{-1}[1 - p(\Theta)]$ and $\Phi^{-1}[\cdot]$ denotes the inverse of the standard normal cumulative probability function. These values can be seen as the failure probability and reliability index conditioned on the set of model parameters. As random variables, $P$ and $B$ have probability distribution functions as well as statistics, such as means and variances. In particular, we define the PDF of $P$ as $f_P(p)$, the PDF of $B$ as $f_B(\beta)$, their means as $\mu_P$ and $\mu_B$, respectively, and their standard deviations as $\sigma_P$ and $\sigma_B$, respectively. Note that $\sigma_P$ and $\sigma_B$ are measures of the uncertainties in the estimates of the failure probability and the reliability index, which arise from parameter uncertainties. Since parameter uncertainties are reducible, $\sigma_P$ and $\sigma_B$ represent measures of uncertainty, which can possibly be reduced by gathering additional observational data.

4. Predictive failure probability and reliability index

In Bayesian analysis, the distribution of a random variable that incorporates parameter uncertainties is called the Bayesian or predictive distribution [1–3]. For the vector of random variables $\mathbf{X}$, whose joint PDF has uncertain parameters $\Theta_f$, the predictive distribution is defined as the expectation of the conditional distribution $\hat{f}_X(x|\Theta_f)$ over the outcome space of the uncertain parameters, i.e.,

$$
\hat{f}_X(x) = \int_{\Theta_f} \hat{f}_X(x|\Theta_f) f_{\Theta_f}(\Theta_f) \, d\Theta_f
$$

where $f_{\Theta_f}(\Theta_f)$ is the posterior distribution of the parameters. In the same vein, we define the predictive failure probability as the expectation of the conditional failure probability $P = p(\Theta)$ over the outcome space of the uncertain parameters $\Theta$. Using (4), this can be written as

$$
\tilde{p} = \int_{\Theta} \phi(\Theta) f_{\Theta}(\Theta) \, d\Theta = \int_{\Theta_f} \int_{\Theta_g} \int_{\Omega(x, \epsilon, \Theta_g)} \hat{f}_X(x|\Theta_f) f_E(\epsilon | \sigma) f_{\Theta_f}(\Theta_f) \times f_{\Theta_g}, \sigma(\Theta_f, \sigma) \, d\epsilon d\sigma \, d\Theta_g
$$

where $f_{\Theta}(\Theta) = f_{\Theta_f}(\Theta_f) f_{\Theta_g}(\Theta_g, \sigma)$ is the posterior joint PDF of the set of parameters, where statistical independence between $\Theta_f$ and $\Theta_g, \Sigma$ is assumed. Clearly, $\tilde{p} = \mu_P$ represents the mean of the distribution $f_P(p)$. The corresponding predictive reliability index is obtained from

$$
\tilde{\beta} = \Phi^{-1}(1 - \tilde{p}) .
$$

However, as shown below, $\tilde{\beta}$ is different from the mean reliability index $\mu_g$.

In a decision problem, provided the utility function is a linear function of the failure probability – as in “total cost = initial cost + cost of failure × probability of failure” – the probability measure to be used is the predictive value in (7). In his June 1, 1988 letter, referring to [11], Ditlevsen argued that “the uncertainty about $p_f$ [$P$ in the present notation] should not have influence on decision making”. In essence, given two decision alternatives with identical predictive failure probabilities and identical initial costs and costs of failure, the decision maker should be indifferent between the two alternatives. This result appears somewhat counterintuitive, as with identical costs and mean failure probabilities, one would expect a preference for the case with smaller uncertainty in the failure probability estimate. But simple derivations in Ditlevsen’s letters show that aversion to uncertainty in failure probability is not logical. Nevertheless, knowledge of the extent of uncertainty in the probability estimate (as measured by $\sigma_P$) is useful for several reasons. Chief among these is that one then knows the extent of this uncertainty that can possibly be removed by gathering of additional data. As we will shortly see, the failure probability estimate is more likely to decrease than increase with increasing data. This potential benefit should be balanced by the cost of gathering the data. Secondly, for the sake of transparency in communicating risk, it is important that the analyst also reports uncertainty in the estimated failure probability, not only the mean value. This may help a client decide whether additional gathering of data is worthwhile. Finally, as demonstrated in [8] by Ditlensven and more recently in [6] and [13], epistemic uncertainties arising from the estimation of model parameters induce statistical dependence among the estimated states of system components. This dependence can have significant influence on the predictive estimate of system failure probability, particularly for redundant systems.

5. Computational methods

In this section we describe methods for computing the predictive failure probability and reliability index, as well as a measure of the uncertainty in their estimates. A straightforward way to compute predictive failure probability in (7) is to treat all random variables and uncertain parameters together so that the set of random variables of the problem is $(\mathbf{X}, \mathbf{E}, \Theta_f, \Theta_g, \Sigma)$. Well known computational methods of structural reliability, such as FORM, SORM or various importance sampling methods [9], can be used for this purpose. These methods typically require a transformation of the random variables into standard normal space. For the present problem the conditional distribution of $\mathbf{X}$ given $\Theta_f$ and the conditional distribution of $\mathbf{E}$ given $\Sigma$ are available, along with the posterior distributions of $\Theta_f$ and $(\Theta_g, \Sigma)$. Because of the conditional form of these distributions, use of the so-called “Rosenblatt” transformation [12] is necessary for solving the reliability problem.

A second approach for computing the predictive failure probability is to first carry out integrations on $\Theta_f$ and $\sigma$ in (7). For this purpose we replace $f_{\Theta_g}, \Sigma(\Theta_f, \sigma)$ by $f_{\Sigma}(\sigma | \Theta_f) f_{\Theta_g}(\Theta_g)$ and note that the integral of the product $f_X(x|\Theta_f) f_{\Theta_f}(\Theta_f)$ over $\Theta_f$ gives the predictive distribution of $\mathbf{X}$ and the integral of $f_E(\epsilon | \sigma) f_{\Theta_g}(\sigma | \Theta_g)$ over $\sigma$ gives the predictive distribution of $\mathbf{E}$ conditioned on $\Theta_g = \Theta_g$. The result is the expression

$$
\tilde{p} = \int_{\Theta_g} \int_{\Omega(x, \epsilon, \Theta_g)} \hat{f}_X(x) \hat{f}_E(\epsilon | \Theta_g) f_{\Theta_g}(\Theta_g) \, d\epsilon d\Theta_g .
$$
The set of random variables of the problem now is \((\mathbf{X}, \mathbf{E}, \Theta)\). This reduction in dimensionality of the problem, however, is achieved at the expense of having to derive predictive distributions \(f_{\tilde{g}}(\mathbf{x})\) of \(\mathbf{X}\) and \(f_E(e|\Theta)\) of \(\mathbf{E}\). In many cases closed-form expressions of these predictive distributions are available (see, e.g., [1,2]). This second formulation is useful for such cases.

The third approach is based on a nested reliability formulation originally suggested by Wen and Chen [14]. Consider the auxiliary structural reliability problem defined by the limit-state function

\[
\tilde{g}(u, \theta) = u + \beta(\theta)
\]

where \(u\) is the outcome of a standard normal random variable \(U\) having the PDF \(\phi(u) = (2\pi)^{-1/2} \exp(-u^2/2)\) and \(\beta(\theta)\) is the conditional reliability index of the original problem for \(\Theta = \theta\). \(U\) is statistically independent of \(\Theta\). The solution of this problem is identical to the predictive failure probability in (7). The proof is as follows:

\[
\begin{align*}
\Pr\left[\tilde{g}(U, \Theta) \leq 0\right] &= \int_{\tilde{g}(u, \theta) \leq 0} \phi(u) f_\Theta(\theta) du d\theta \\
&= \int_{\tilde{g}(u, \theta) \leq 0} \int_{-\infty}^{\tilde{g}(u, \theta)} \phi(u) du f_\Theta(\theta) d\theta \\
&= \int_{\tilde{g}(u, \theta) \leq 0} \Phi[\tilde{g}(u, \theta)] f_\Theta(\theta) d\theta \\
&= \int_{\tilde{g}(u, \theta) \leq 0} p(\theta) f_\Theta(\theta) d\theta \\
&= \tilde{p}.
\end{align*}
\]

It follows that the predictive failure probability can be computed by solving the problem in (10), which in turn requires the solution of the conditional reliability problem defined by (4) — hence the reason for the name nested reliability analysis. Obviously this approach is convenient if the conditional problem in (4) has a closed-form solution expressed in terms of the parameters \(\Theta\). However, even when such a closed-form solution is not available, the above formulation offers a useful approximation for the predictive reliability index. If we assume the uncertain reliability index \(B = \beta(\Theta)\) has a normal distribution, then the limit-state function in (10) is a linear function of two independent normal random variables, \(U\) and \(B\). The solution of the corresponding reliability index then is simply the mean \(\mu_t = 0 + \mu_B = \mu_B\) of the limit-state function divided by its standard deviation \(\sigma_t = \sqrt{1 + \sigma_B^2}\).

It follows that the predictive reliability index is approximately given by

\[
\bar{\beta} = \frac{\mu_B}{\sqrt{1 + \sigma_B^2}}. \tag{12}
\]

The only approximation involved in this derivation is the assumption of normal distribution of the uncertain reliability index. In many cases this assumption is not too far from reality. It is interesting to see that the predictive reliability index increases with decreasing variance of the uncertain reliability index, provided the mean reliability index is positive and remains unchanged. Furthermore, the predictive reliability index is found to be smaller than the mean reliability index in absolute value.

The use of (12) requires knowledge of the mean and standard deviation of the conditional reliability index. These are not easy to compute exactly, since they require knowledge of the distribution \(f_B(\beta)\). However, they can be easily computed by applying the first-order approximation method to the function \(B = \beta(\Theta)\). The result is

\[
\begin{align*}
\mu_B &\approx \beta(M_{\Theta}) \tag{13} \\
\sigma_B^2 &\approx (\nabla_\Theta \beta)^T M_{\Theta} \nabla_\Theta \beta \quad \Theta = M_{\Theta} \tag{14}
\end{align*}
\]

where \(M_{\Theta}\) and \(\Sigma_{\Theta}\) are the posterior mean vector and covariance matrix of \(\Theta\), respectively, and \((\nabla_\Theta \beta)_{\Theta = M_{\Theta}}\) is sensitivity vector of the conditional reliability index with respect to parameters \(\Theta\) evaluated at the posterior mean values. It is seen that a single solution of the conditional reliability problem in (4) at the posterior mean values of the parameters, along with parameter sensitivity analysis is sufficient to approximate the mean and variance of the conditional reliability index from (13), (14) and the predictive reliability index according to the approximate formula in (12). The predictive failure probability, \(\bar{p}\), is computed by the inverse of (8).

As mentioned earlier, it is useful to have a measure of the uncertainty in the failure probability estimate, which arises from model parameter uncertainties. One way to express this uncertainty is to provide Bayesian credibility bounds, which express bounds within which the failure probability lies with a specified probability. Since, roughly speaking, the interval \((\mu_B \pm \sigma_B)\) contains 70% probability, the 70% credibility interval on the uncertain failure probability is approximately given by

\[
\langle P \rangle_{70\%} \approx \Phi[-(\mu_B + \sigma_B)], \Phi[-(\mu_B - \sigma_B)]. \tag{15}
\]

This interval is a measure of the uncertainty in the failure probability estimate, which can possibly be reduced by gathering of additional data.

Although it is possible to derive expressions similar to (13), (14) for the mean and standard deviation of the uncertain failure probability \(P = p(\Theta)\), the resulting expressions usually are not as accurate due to the strong skewness of the distribution \(f_P(p)\). Furthermore, for small failure probabilities, the one-standard deviation lower credibility bound of the uncertain failure probability often has a negative value due to the highly skewed nature of the distribution. The expression in (15) avoids these shortcomings.

6. Illustrative example

Consider a structural component having the limit-state model

\[
g(x) = x_1 - x_2 \quad \tag{16}
\]

where \((x_1, x_2)\) is the outcome of the vector of random variables \(\mathbf{x} = (X_1, X_2)\). Assume \(X_1\) and \(X_2\) are statistically independent normal random variables with unknown means \(M_1\) and \(M_2\),...
and known variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Suppose the available information for estimating the mean values are sample observations of size $n$ of $X_1$ and $X_2$ with respective sample means $\bar{x}_1$ and $\bar{x}_2$. Assuming independence of $M_1$ and $M_2$ and using non-informative priors, the posterior distributions of $M_1$ and $M_2$ are found to be independent normals with means $\bar{x}_1$ and $\bar{x}_2$ and standard deviations $\sigma_1/\sqrt{n}$ and $\sigma_2/\sqrt{n}$, respectively. Thus, in this problem we have a precisely known limit-state model and uncertain distribution parameters $\Theta = (M_1, M_2)$. Due to the linear form of the limit-state function and the normal distribution of the random variables, the conditional reliability index is easily found to be

$$ B = \beta(\Theta) = \frac{M_1 - M_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}. $$

(17)

Because $B$ is a linear function of the random variables $M_1$ and $M_2$, one can easily determine its mean and variance as

$$ \mu_B = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} $$

(18)

$$ \sigma_B = \frac{1}{\sqrt{n}}. $$

(19)

Furthermore, since $M_1$ and $M_2$ are normally distributed, $B$ has the normal distribution. It follows that in this case (12) gives an exact result and the predictive reliability index is given by

$$ \tilde{\beta} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2 + \sigma_2^2} \sqrt{1 + 1/n}}. $$

(20)

It is seen that, provided the sample means remain constant, the predictive reliability index increases in absolute value with increasing sample size $n$ and approaches the mean reliability index as $n$ approaches infinity. Of course additional data may affect the sample means and, therefore, the predictive reliability index. However, since $\mu_B$ is the median reliability index and $\tilde{\beta} < \mu_B$, it is more likely that the reliability index will increase than decrease with increasing sample size. This is consistent with the “remunerability” requirement defined in [4,5], as mentioned in the prologue of this paper.

Now suppose the limit-state model includes a residual term $\varepsilon$,

$$ g(x, \varepsilon) = x_1 - x_2 + \varepsilon $$

(21)

where $\varepsilon$ denotes the outcome of a normal random variable $\varepsilon$ having a zero mean and an unknown variance $\Sigma$. (Such limit-state models for seismic fragility of electrical substation equipment have been developed in [13].) Suppose the posterior mean of $\Sigma$ is a fraction $r$ of $\sigma_1^2 + \sigma_2^2$, i.e., $\mu_\Sigma = r(\sigma_1^2 + \sigma_2^2)$, and its posterior coefficient of variation is $\delta_\Sigma$. The set of uncertain parameters now is $\Theta = (M_1, M_2, \Sigma)$ and the conditional reliability index is given by

$$ B = \beta(\Theta) = \frac{M_1 - M_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + \Sigma}}. $$

(22)

The above is no more a linear function of uncertain parameters $\Theta$. Therefore, exact solutions of the mean and variance are not easy to obtain. If the first order approximations in (13), (14) are used, the results are

$$ \mu_B \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2 + \sigma_2^2} \sqrt{1 + 1/r}} $$

(23)

$$ \sigma_B \approx \frac{1}{\sqrt{n(1 + r)}} + \frac{(\bar{x}_1 - \bar{x}_2)^2}{\sigma_1^2 + \sigma_2^2} \frac{1}{4(1 + r)} \left( \frac{r \delta_\Sigma}{1 + r} \right)^2. $$

(24)

Using these relations in (12), the corresponding approximate estimate of predictive reliability is obtained as

$$ \tilde{\beta} \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2 + \sigma_2^2} \sqrt{1 + 1/r + 1/n + (\bar{x}_1 - \bar{x}_2)^2 \left( \frac{r \delta_\Sigma}{2(1 + r)} \right)^2}}. $$

(25)

As in the previous case, provided the sample means $\bar{x}_1$ and $\bar{x}_2$ remain constant, the predictive reliability index increases with decreasing posterior coefficient of variation of $\Sigma$.

Fig. 2 compares the above approximation of the predictive reliability index for $(\bar{x}_1 - \bar{x}_2)/\sqrt{\sigma_1^2 + \sigma_2^2} = 3$, $r = 0.2$ and $n = 10$ and 20 as a function of the coefficient of variation $\delta_\Sigma$ with “exact” results obtained by simulation with a 3% coefficient of variation. Also shown in the figure are approximations of the predictive reliability index obtained by applying first- and second-order reliability methods (FORM and SORM) to the formulation in (9).
Note that the approximation in (25) only involves the first and second moments of uncertain parameters. On the other hand, simulation and FORM/SORM results require the full distribution of uncertain parameters. In the present case, $M_1$ and $M_2$ are Gaussian distributed and $\Sigma$ is assumed to be lognormally distributed. It is seen that the approximation in (12) is in close agreement with the simulation results. In particular, it correctly predicts the decrease in the predictive reliability index with increasing $\delta_\Sigma$ or decreasing sample size (with fixed values of $\bar{x}_1, \bar{x}_2$ and $r$). Ironically, the FORM approximation predicts an increase in the predictive reliability index with increasing coefficient of variation of $\Sigma$, and the SORM approximation shows essentially no variation with $\delta_\Sigma$. It is believed that the poor performance by FORM and SORM is due to nonlinearity in the dependence of the conditional reliability index on the uncertain parameter $\Sigma$, as is evident in (22).

Next consider a $K$-out-of-$N$ system with exchangeable components, each characterized by the limit-state function in (21). Such a system survives if $K$ or more of its $N$ components survive. Conditional on the parameters $\Theta = (M_1, M_2, \Sigma)$, random variables $X_1, X_2$ and $E$ for different components are statistically independent. However, all components share the same set of uncertain parameters $\Theta = (M_1, M_2, \Sigma)$. As discussed in [6,13], this type of common parameter uncertainty induces statistical dependence among the estimated states of system components, which can have a significant influence on the predictive reliability of the system.

Owing to the conditional independence of the component states, the conditional probability of failure of the system...
for given $\Theta = (M_1, M_2, \Sigma)$ is described by the binomial cumulative distribution function

$$p(\Theta) = \sum_{i=N-k+1}^{N} \frac{N!}{i!(N-i)!} [q(\Theta)]^i [1 - q(\Theta)]^{N-i}$$

(26)

where

$$q(\Theta) = \Phi \left( -\frac{M_1 - M_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + \Sigma}} \right)$$

(27)

is the conditional failure probability of each component. The corresponding conditional reliability index is given by $\beta(\Theta) = \Phi^{-1} [1 - p(\Theta)]$. Using this expression, the predictive reliability index is computed by the nested reliability method employing the limit-state function in (10). An approximate estimate of the predictive reliability index is obtained by use of the formula in (12) together with first-order approximations of the mean and standard deviation of the reliability index as in (13), (14). Note again that the latter analysis only requires the first and second moments of the parameters and computation of the conditional reliability index and its sensitivities for mean values of the parameters. On the other hand, computation of the predictive reliability index by nested reliability analysis employing FORM, SORM or Monte Carlo simulation requires the full distribution of the parameters.

Fig. 3 compares approximate estimates of the predictive reliability index with FORM, SORM and “exact” simulation results for various systems with $(\bar{x}_1 - \bar{x}_2)/\sqrt{\sigma_1^2 + \sigma_2^2} = 3$, $r = 0.2$, $n = 10$ and 20, and as functions of the coefficient of variation $\delta_2$. Considered are systems having $N = 5$ components with $K = 1$ (parallel systems), $K = 3$, and $K = 5$ (series systems). For redundant systems with $K = 1$ and 3, where the reliability index is higher than 3, importance sampling around the design point is used. All simulation results have less than 3% coefficient of variation. It is observed that in all cases the predictive reliability index of the system decreases with increasing coefficient of variation $\delta_2$ and decreasing sample size $n$ (for fixed values of $\bar{x}_1$, $\bar{x}_2$ and $r$). Furthermore, parameter uncertainties have a more profound effect on redundant systems ($K < N$). As noted in [6,13], this is due to the effect of dependence among estimated states of system components, which arises from parameter uncertainties that are common to all components. The simple approximation in (12) provides close agreement with simulation results for $1 < K$ and fair approximation for $K = 1$ (parallel systems). It is believed that the lack of close agreement for parallel systems is due to the approximation involved in computing the mean and variance from (13), (14) rather than the approximation inherent in deriving (12). Overall, the expression in (12) is found to provide a consistent and reasonably accurate estimate of the predictive reliability index. It is worth emphasizing once again that this equation requires nothing more than reliability and sensitivity analyses with posterior mean values of the parameters together with knowledge of the posterior covariance matrix of the uncertain parameters. With its simplicity, this formula provides an attractive alternative for practical structural reliability problems involving uncertain parameters, as well as for probabilistic codified design.

7. Summary and conclusions

The problem of structural reliability under parameter uncertainties is considered. The parameters are involved in probabilistic and physical models, which are used to formulate the reliability problem. In practice, these parameters must be estimated by statistical inference using observational data – a process, which invariably yields uncertainties. The measure of reliability that incorporates parameter uncertainties is termed the predictive reliability index. Methods for computing the predictive reliability index and the corresponding failure probability are described. One method leads to a simple approximation formula, which involves a single computation of the conditional reliability index and its parameter sensitivities with respect to mean values of the parameters. This formulation also provides a measure of the uncertainties in the estimated reliability index and failure probability, which arise from parameter uncertainties. An illustrative example demonstrates the computation of the predictive reliability index and the accuracy of the simple formula for component and system reliability problems.

References


