Bayesian networks for system reliability reassessment

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Abstract

This paper proposes a methodology to apply Bayesian networks to structural system reliability reassessment, with the incorporation of two important features of large structures: (1) multiple failure sequences, and (2) correlations between component-level limit states. The proposed method is validated by analytical comparison with the traditional reliability analysis methods for series and parallel systems. The Bayesian network approach is combined with the branch-and-bound method to improve its efficiency and to facilitate its application to large structures. A framed structure with multiple potential locations of plastic hinges and multiple failure sequences is analyzed to illustrate the proposed method. © 2001 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Bayesian network; Reliability reassessment; System reliability; Failure sequence

1. Introduction

Reliability-based design and in-service inspection (or development testing) are two complementary approaches for ensuring the safe performance of structures. Physics-based computational reliability methods are much more cost-efficient for large and complex structures, and are tools that could be used when experimental data are difficult to obtain. However, due to the approximations in the computational model and the limited statistical data on the input variables, there may be uncertainty or error in this computation. Therefore, when inspection/test information is available, the results of the two approaches should be combined for a robust re-estimation of the structural reliability. Bayes’ theorem, which is used for probabilistic updating, provides an appropriate framework for this purpose. Several important studies have been reported to incorporate the information from inspection for fatigue reliability reassessment using Bayes’ theorem [1–4]. All these studies are based on component level reliability analysis and individual limit states.

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In real engineering applications for complicated structures with multiple components or multiple failure mechanisms, system reliability needs to be evaluated. Many enumeration and simulation-based techniques for system reliability analysis have been developed in the past two decades [5–9]. This paper focuses on situations where new information on component performance or system performance may become available through testing or inspection after analysis, and explores techniques to update the prior reliability prediction using this information. Several studies have been proposed to update system level reliability estimation when system level test data are available, also using Bayes’ theorem [10–12]. In these studies, the system is viewed essentially through a single limit state. Depending on the system configuration, a system test resulting in a success does not necessarily imply that all components in the system were successful, nor does a system test failure necessarily imply that all components have failed. Therefore, Martz and Waller [13] concluded that system test data usually provide no information on component performance.

In recent years, a Bayesian network methodology has been developed in the field of artificial intelligence [14]. This causal network is an inference engine for the calculation of beliefs or probability of events given the observation/evidence of other events in the same network. Bayesian networks have been applied mostly in the field of artificial intelligence. Recently, Bayesian networks have been used in engineering decision strategy [14], in the safety assessment of software-based systems [15], and in model-based adaptive control [16,17]. It has also been proposed to apply Bayesian networks to the risk assessment of water distribution systems, as an alternative to fault tree analysis [18–20].

Due to its ability of uncertainty propagation and updating through the nodes/components in the network, the Bayesian network overcomes the hurdle of earlier Bayesian updating methods where information from system-level could not be transformed to component level. Therefore, it should provide a promising framework for system reliability reassessment. However, so far it has not been used in the system reliability re-analysis of mechanical or civil structures.

In this paper, the difficulties in the application of Bayesian networks to structural system reliability reassessment are investigated and solved through several modifications to the Bayesian network methods currently used in literature. Two important features of structural systems are incorporated: (1) multiple failure sequences, and (2) correlations between component-level limit states. The proposed method is validated by analytically comparing with the traditional reliability analysis methods for series and parallel systems. Also, the Bayesian network approach is combined with the branch-and-bound method to increase its efficiency and to facilitate its application to large structures. A framed structure with multiple potential locations of plastic hinges and multiple failure sequences is analyzed to illustrate the proposed approach.

2. Bayesian networks

A Bayesian network (BN), also known as a Bayesian belief network (BBN), is a directed acyclic graph (DAG) formed by the variables (nodes) together with the directed edges, attached by a table of conditional probabilities of each variable on all its parents [14]. Therefore, it is a graphical representation of uncertain quantities (and decisions) that explicitly reveals the probabilistic causal dependence between the variables as well as the flow of information in the model.
In a Bayesian network, the nodes without any arrows directed into them are called root nodes and they have prior probability tables (discrete nodes) or functions (continuous nodes) associated with them. The nodes that have arrows directed into them are called child nodes and the nodes that have arrows directed from them are called parent nodes. Each child has a conditional probability table (or function) associated with it, given the state (or value) of the parent nodes. Consider a Bayesian network over \( U = \{X_1, \ldots, X_n\} \), where \( X_1, \ldots, X_n \) are the nodes. Then, based on the chain rule, the joint probability \( p(X_1, \ldots, X_n) \) is

\[
p(U) = p(X_1, \ldots, X_n) = \prod_{i=1}^{n} p(X_i | \pi_i)
\]

where \( \pi_i \) is the set of parents of node \( X_i \). The marginal probability of \( X_i \) is:

\[
p(X_i) = \sum_{\text{except } X_i} p(U)
\]

A Bayesian network’s main application is as an inference engine for the calculation of beliefs of events given the observation of other events, called evidence. For a Bayesian network, this task consists of the calculation of the probability of the occurrence of some events given the evidence. Assume an evidence \( e \) is found, we have

\[
p(U | e) = \frac{p(U, e)}{p(e)} = \frac{p(U, e)}{\sum_U p(U, e)}
\]

As to the elicitation of probability and conditional probability at the nodes and edges, these may be expressed either as discrete numbers or as a continuous function (corresponding to summation or integration operations, respectively). In previous studies, the Bayesian network computation has been implemented only for the discrete case, and the modeling is performed with a commonly used software package HUGIN [14]. For the continuous case, the state of the variable may be discretized into a finite number of states/values.

The Bayesian network methodology has been developed and applied mostly in the field of artificial intelligence. Most studies have focused on developing efficient algorithms for large Bayesian network construction and computation [21–23]. Fig. 1 is a typical Bayesian network for quality assessment [15]. In this example, “production quality” is the target node. The probability or conditional probability for each node is assigned based on experience and judgment. Through uncertainty propagation and updating in Eqs. (1)–(3), information on some observable nodes may be used for the assessment of target nodes.

Compared to fault tree analysis of system reliability, Bayesian network avoids duplicating nodes for common cause analyses. Consider the example of probability risk assessment of a power distribution system, with both fault tree and Bayesian network illustrated [18] in Figs. 2 and 3, respectively. In system failure analysis, each variable has two states: failure denoted by 1 and no failure denoted by 0. \( Q \) represents system failure.

Using Eq. (1), the joint probability corresponding to the Bayesian network in Fig. 3 is factorized as:
Fig. 1. Bayesian Network for the node “system quality”.

Fig. 2. Fault tree.

Fig. 3. Bayesian network corresponding to the fault tree.
\[ p(X) = p(a)p(b)p(c)p(d|a, b, c)p(f)p(g)p(e|d, f, g)p(h|b, f, g) \]
\[ \times p(i)p(j|h, i)p(k)p(l|e, k)p(m)p(n|l, m)p(q|j, n) \]  
(4)

The system failure probability is

\[ p(Q = 1) = \sum_{a=0}^{1}\sum_{b=1}^{1}\cdots\sum_{n=0}^{1}p(a)p(b)p(c)p(d|a, b, c)p(f)p(g)p(e|d, f, g) \]
\[ \times p(h|b, f, g)p(i)p(j|h, i)p(k)p(l|e, k)p(m)p(n|l, m)p(q = 1|j, n) \]  
(5)

If a system failure is observed, then the marginal probability of all the nodes in the network can be updated as

\[ p(X|Q = 1) = \frac{p(X, Q = 1)}{p(Q = 1)} \]  
(6)

In the fault tree or event tree-based traditional system reliability analysis, when system level test data are available, we can only determine the posterior probability of system failure by using Bayes’ theorem. In this case, the system is viewed essentially as a single component (i.e. with a single limit state), and it is not possible to update component performance statistics [13]. However, using the Bayesian network, component performance can also be updated using the system test data, as in Eq. (6). In fact, the test data on any node of the network provides information on all other nodes, which can be updated using the Bayesian network.

3. Application of Bayesian networks to civil and mechanical system reliability analysis

Though the Bayesian network is attractive due to the possibility of system reliability updating with newly available evidence, it has not been used in the system reliability analysis in mechanical or civil structures. This paper studies this problem and explores the possibility of applying Bayesian networks for structural system reliability assessment and updating.

Two problems are encountered in the application of Bayesian networks (as currently used in the literature) to the reliability reassessment of mechanical or civil systems: (1) correlations among component failures, and (2) the existence of multiple failure sequences. This paper develops methods to overcome both these hurdles, as discussed below.

3.1. Incorporation of correlation among component limit states

Consider the two-bar system in Fig. 4. Consider it as a series system first, in which the system is defined to fail when any one of the bars fails. Following a previous application of Bayesian networks in the risk assessment of water distribution systems [18–20], it is natural to construct a three-node Bayesian network, with two component failures \( A \) and \( B \), and system failure \( C \) as in Fig. 5. The conditional probability \( P(C|A, B) \) is shown in Table 1, in which value 1 represents failure and 0 represents survival. However, using the traditional view of Bayesian network, this
gives the impression that \( A \) and \( B \) are independent nodes and that there is no link between them. However, in this case, there is dependence between the failures of bar \( A \) and bar \( B \). One kind of dependence is the statistical correlation between the states of \( A \) and \( B \) due to the common random variables in their limit states. That means \( A \) and \( B \) are not independent even if they fail at the same time, which is denoted as: \( P(AB) \neq P(A)P(B) \). However, the construction of Bayesian network in Fig. 5 and the use of Eq. (1) lead to the result \( P(AB) = P(A)P(B) \), since according to the definition of Bayesian network, \( A \) and \( B \) in Fig. 5 are root nodes, implying they are independent.

To further illustrate this problem, using the algorithm of Eqs. (1) and (2), the series system failure probability obtained from the Bayesian network in Fig. 5 and Table 1 is:

\[
P(C = 1|A, B) = P(A = 1)P(B = 1) + P(A = 1)P(B = 0) + P(A = 0)P(B = 1)
\]

We may simply write the result in Eq. (7) as \( P(A) + P(B) - P(A)P(B) \). Since it is known that for this series system, the failure probability is actually \( P(A) + P(B) - P(A, B) \), Eq. (7) is only correct when \( A \) and \( B \) are independent, which is not the case for this example. Therefore, the construction in Fig. 5 needs to be modified to incorporate the dependence due to correlated limit states.

To solve this problem, a new Bayesian network is constructed with all the input random variables as additional, but root nodes. For example, consider \( R_A \), the strength of \( A \), \( R_B \), the strength of

\[
\begin{array}{c|c|c}
\text{Table 1} \\
\hline
\text{given the states of components } A \text{ and } B \text{ in a series system} & A = 1 & A = 0 \\
\text{ } & 1 & 0 \\
B & 1 & 1 & 0 \\
B = 1 & 1 & 0 & 0 \\
B = 0 & 0 & 0 & 0 \\
A = 1 & 0 & 0 & 0 \\
\text{ } & 0 & 0 & 0 \\
\hline
\end{array}
\]
$B$, and the applied load $W$ to be the random variables in this problem. It is obvious that the failure probability of $A$ is a function of $R_A$ and $W$, and the failure probability of $B$ is a function of $R_B$ and $W$. [Also, the joint probability $P(A, B)$ can be computed as discussed in the next section]. Then we can construct the Bayesian network as in Fig. 6. Fig. 6(a) corresponds to the case of uncorrelated variables, and Fig. 6(b) corresponds to the case of correlated variables. The series system example is continued in Section 3.3.1. with this new network.

It is clear from Fig. 6(b) that when some of the input random variables are correlated, we can put the correlated random variables in one root node, associated with their joint probability. In this case, the root node represents a vector of variables, not a single variable.

![Bayesian network](image)

(a) Uncorrelated variables  
(b) Correlated variables

Fig. 6. Bayesian network accounting for the dependence of the two bars; (a) uncorrelated variables, (b) correlated variables.

### 3.2. Incorporation of multiple failure sequences

Now consider the 2-bar system in Fig. 4 as a parallel system, i.e. the system fails only when both bars $A$ and $B$ fail. In this case, besides the statistical dependence considered in the previous subsection, sequential or causal dependence between $A$ and $B$ also needs to be incorporated. This sequential dependence means that the failure of one bar will change the failure probability of another bar. There are two different failure sequences for this system: $A$ fails first followed by $B$, and $B$ fails first followed by $A$. Since the Bayesian network does not allow cycles, we can not have $A \rightarrow B$.

To account for multiple failure sequences, we can still construct a Bayesian network with $A$ and $B$ both parents of $C$ as shown in Fig. 6, but with additional information on sequential failure probabilities represented in the conditional probability table in Table 2.

<table>
<thead>
<tr>
<th>$A = 1$</th>
<th>$A = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = 1$</td>
<td>$P(A = 1</td>
</tr>
<tr>
<td>$B = 0$</td>
<td>$P(B = 1</td>
</tr>
</tbody>
</table>
In Table 2, $A = 1$ and $B = 1$ represent the failures of bars $A$ and $B$ starting from an intact structure, $A(B)$ represents the updated state of $A$ after $B$’s failure, and $B(A)$ represents the updated state of $B$ after $A$’s failure. As an example, the relation $P(C = 1 | A = 1, B = 0) = P(B(A) = 1 | A = 1, B = 0)$ means that the probability that $C = 1$ will occur given $A$ fails “first” is equal to $B$’s failure probability given $A$ fails “first”, since $C = 1$ is defined when both $A$ and $B$ fail. Through this construction of the conditional probability table for $P(C = 1 | A, B)$, the problem of multiple failure sequences and the effect of the failure of one bar on another can be solved and incorporated in a Bayesian network.

Therefore, using the proposed approach, both types of dependence (statistical and sequential) of the failure of the two bars are adequately incorporated in a Bayesian network. The statistical correlation between the states of the components is incorporated by introducing the input random variables into the Bayesian network and taking them as root nodes. Sequential effects are incorporated through the definition of conditional system failure probability given a component failure sequence.

3.3. Implementation of system reliability analysis using the Bayesian network

3.3.1. Series system

Based on Eqs. (1) and (2), the probability of system failure in Fig. 6 for the series system can be obtained as:

$$P(C = 1) = \sum_{A=0}^{1} \sum_{B=0}^{1} \int_{r_A} \int_{r_B} \int_{w} P(C = 1 | A, B)P(A | r_A, w)P(B | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw$$

$$= \int_{r_A} \int_{r_B} \int_{w} P(A = 0 | r_A, w)P(B = 1 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw$$

$$+ \int_{r_A} \int_{r_B} \int_{w} P(A = 1 | r_A, w)P(B = 0 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw$$

$$+ \int_{r_A} \int_{r_B} \int_{w} P(A = 1 | r_A, w)P(B = 1 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw$$

(8)

where $f(r_A), f(r_B)$ and $f(w)$ are the PDFs of the input random variables.

Note that in this example, the three input random variables are taken as independent variables for illustrative purposes. If they are correlated, the products of the three PDFs in the integration in Eq. (8) will be replaced with their joint PDF.

To further illustrate this problem, assume that Fig. 4 is a brittle structure, and each bar takes half of the load $W$. Then we can obtain the following conditional probabilities needed in Eq. (8):

$$P(A = 1 | r_A \leq \frac{W}{2}) = 1, P(A = 0 | r_A \geq \frac{W}{2}) = 1$$

(9a)

and

$$P(B = 1 | r_B \leq \frac{W}{2}) = 1, P(B = 0 | r_B \geq \frac{W}{2}) = 1$$

(9b)
Therefore Eq. (8) is written as:

\[
P(C = 1) = \int_{R_A \geq \frac{w}{2}, R_B \leq \frac{w}{2}} f(r_A)f(r_B)f(w)dr_Adr_Bdw + \int_{R_A \leq \frac{w}{2}, R_B \geq \frac{w}{2}} f(r_A)f(r_B)f(w)dr_Adr_Bdw
\]

\[
+ \int_{R_A \leq \frac{w}{2}, R_B \leq \frac{w}{2}} f(r_A)f(r_B)f(w)dr_Adr_Bdw
\]

(10)

It can be also seen that each of the three terms on the right hand side of Eq. (8) is actually the limit state-based joint probability of two correlated events, i.e.

\[
P(C = 1) = P(A = 0, B = 1) + P(A = 1, B = 0) + P(A = 1, B = 1)
\]

\[
= P\left( R_A \geq \frac{w}{2}, R_B \leq \frac{w}{2} \right) + P\left( R_A \leq \frac{w}{2}, R_B \geq \frac{w}{2} \right) + P\left( R_A \leq \frac{w}{2}, R_B \geq \frac{w}{2} \right)
\]

(11)

The three terms in Eq. (11) can be calculated separately using system reliability methods (second order bounds [24], multi-normal integration [25], or Monte Carlo simulation) without complete numerical integration.

Rewriting Eq. (11) as:

\[
P(C = 1) = P(A = 1, B = 0) + P(A = 1, B = 1) + P(A = 0, B = 1)
\]

\[
= P(A = 1) + P(B = 1) - P(A = 1, B = 1)
\]

\[
= P\left( R_A \leq \frac{w}{2} \right) + P\left( R_B \leq \frac{w}{2} \right) - P\left( R_A \leq \frac{w}{2}, R_B \leq \frac{w}{2} \right)
\]

(12)

This is in the same form as derived by the usual series system reliability analysis, which demonstrates the validity of the new construction in Fig. 6 of the Bayesian network for this series system. Each term on the right hand side can be calculated using the reliability methods mentioned above.

3.3.2. Parallel system

Consider again the brittle structure in Fig. 4, where each bar carries half of the load \( W \). In this case, if one bar fails, it is removed and subsequently the other bar will carry the entire load, and the system failure is defined to occur when both bars fail. The limit state-based reliability method in the previous subsection may also be used here for the computational implementation. In addition to the limit states for \( A = 1 \) and \( B = 1 \) for the intact structure, the updated failure events of \( A(B) \) (\( A \)'s failure after \( B \)'s failure), and \( B(A) \) (\( B \)'s failure after \( A \)'s failure) also need to be defined to account for multiple failure sequences:

\[
A(B) = 1 \Rightarrow R_A \leq w
\]

\[
B(A) = 1 \Rightarrow R_B \leq w
\]

(13)

Therefore, combining the Bayesian network in Fig. 6, the conditional probability table in Table 2, and the limit state for each component, we obtain
\[ P(C = 1) = P(C = 1|A = 1, B = 1) \times P(A = 1, B = 1) + P(C = 1|A = 0, B = 1) \times P(A = 0, B = 1) \\
+ P(C = 1|A = 1, B = 0) \times P(A = 1, B = 0) + P(C = 1|A = 0, B = 0) \times P(A = 0, B = 0) \]
\[ = P(R_A \leq \frac{w}{2} \cap R_B \leq \frac{w}{2}) + P(R_A \leq \frac{w}{2} \cap R_B \geq \frac{w}{2}) \times P(R_A \geq \frac{w}{2} \cap R_B \leq \frac{w}{2}) \]
\[ + P(R_B \leq \frac{w}{2} | R_A \leq \frac{w}{2} \cap R_B \geq \frac{w}{2}) \times P(R_A \leq \frac{w}{2} \cap R_B \geq \frac{w}{2}) \]
\[ = P(R_A \leq \frac{w}{2} \cap R_B \leq \frac{w}{2}) + P(R_A \leq w \cap R_A \geq \frac{w}{2} \cap R_B \leq \frac{w}{2}) \]
\[ + P(R_B \leq w \cap R_A \leq \frac{w}{2} \cap R_B \geq \frac{w}{2}) \]  

To compare this result with the traditional reliability approach, consider the event tree method to calculate the failure probability. The two failure sequences are represented by the following intersections:

\[ A \rightarrow B : R_A \leq \frac{w}{2} \cap R_B \leq w \]
\[ B \rightarrow A : R_B \leq \frac{w}{2} \cap R_A \leq w \]

The system failure probability is the union of the two failure sequences:

\[ P_f = P \left[ \left( R_A \leq \frac{w}{2} \cap R_B \leq w \right) \cup \left( R_B \leq \frac{w}{2} \cap R_A \leq w \right) \right] \]
\[ = P \left[ \left( R_A \leq \frac{w}{2} \cap \left( R_B \leq \frac{w}{2} \cup \left( R_A \leq \frac{w}{2} \cap R_B \leq w \right) \right) \right) \cup \right] \]
\[ \left( R_B \leq \frac{w}{2} \cap \left( R_A \leq \frac{w}{2} \cup \left( R_A \leq \frac{w}{2} \cap R_A \leq w \right) \right) \right) \]
\[ = P \left[ \left( R_A \leq \frac{w}{2} \cap R_B \leq \frac{w}{2} \right) \cup \left( R_A \leq \frac{w}{2} \cap R_B \leq w \right) \cup \left( R_B \leq \frac{w}{2} \cap R_A \leq w \right) \right] \]  

Since the three events of the union in the final step in Eq. (15) are mutually exclusive, we have

\[ P_f = P(R_A \leq \frac{w}{2} \cap R_B \leq \frac{w}{2}) + P(R_A \leq \frac{w}{2} \cap R_B \leq w) \]
\[ + P(R_B \leq \frac{w}{2} \cap R_A \leq w) \]  

It can be seen that the system failure probability obtained by the Bayesian network in Eq. (14) and that calculated using the traditional system reliability method provide the exactly same result for this problem.

Thus the proposed modifications to the traditional Bayesian network make it consistent for application to structural system reliability analysis. However, the advantage of this approach is not significant in system reliability prediction (forward propagation). Traditional fault-tree and event-tree methods can already predict system reliability. (In fact, the Bayesian network needs more computational effort to establish the conditional probability table, as discussed later in Section 4). The real advantage of the Bayesian network approach is in reliability reassessment (backward propagation), when new data becomes available. This is investigated in the next section.
3.4. Reliability updating using the Bayesian network

When new information on one or more nodes is available, the probability of all the other nodes and the whole system can be updated using the Bayesian network. Eqs. (11) and (14) deal with forward propagation, i.e. computation of system failure probability from input random variable statistics. The Bayesian network allows backward propagation to update the probabilistic information of any node given evidence on component or system performance.

Consider the series system in Fig. 4 for example. If a system failure is observed, the failure probability of Bar A or Bar B and the probability distributions of the corresponding random variables \( R_A, R_B, \) and \( W \), can be updated as:

\[
P(A = 1 | C = 1) = \frac{P(A = 1, C = 1)}{P(C = 1)} = \frac{P(R_A \leq \frac{w}{2})}{P(C = 1)}
\]

\[
P(B = 1 | C = 1) = \frac{P(B = 1, C = 1)}{P(C = 1)} = \frac{P(R_A \leq \frac{w}{2})}{P(C = 1)}
\]

\[
f(r_A | C = 1) = \frac{dF(r_A | C = 1)}{dr_A} = \frac{d}{dr_A} \left( \frac{P(R_A \leq r_A, C = 1)}{P(C = 1)} \right)
\]

where \( P(C = 1) \) has been given in Eq. (11). The updated CDF of \( R_A, P(R_A \leq r_A, C = 1) \), can be obtained as follows, using the theorem in Eq. (2):

\[
P(R_A \leq r_A, C = 1) = \int_{-\infty}^{r_A} \int_{r_B}^{r_A} P(A = 0 | r_A, w)P(B = 1 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw
\]

\[
+ \int_{-\infty}^{r_A} \int_{r_B}^{r_A} P(A = 1 | r_A, w)P(B = 0 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw
\]

\[
+ \int_{-\infty}^{r_A} \int_{r_B}^{r_A} P(A = 1 | r_A, w)P(B = 1 | r_B, w)f(r_A)f(r_B)f(w)dr_A dr_B dw
\]

\[
= P \left( R_A \geq \frac{w}{2} \cap R_B \leq \frac{w}{2} \cap R_A \leq r_A \right) + P \left( R_A \leq \frac{w}{2} \cap R_B \geq \frac{w}{2} \cap R_A \leq r_A \right)
\]

\[
+ P \left( R_A \leq \frac{w}{2} \cap R_B \leq \frac{w}{2} \cap R_A \leq r_A \right)
\]

Similar formulae as in Eqs. (19) and (20) may be applied to update the distributions of the other two variables \( R_B \) and \( W \).

If a failure of bar B (component failure) is observed, then other nodes may be updated as

\[
P(A = 1 | B = 1) = \frac{P(A = 1, B = 1)}{P(B = 1)}
\]
\[ f(r_B | B = 1) = \frac{dF(r_B | B = 1)}{dr_B} = \frac{d}{dr_B}\left(\frac{P(R_B \leq r_B, B = 1)}{P(B = 1)}\right) \]  

(22)

Similar formulae may be applied to update the distributions of \( R_A \) and \( W \).

Information on component failure can also be used to update the system failure probability through the Bayesian network. In the case of the series system example, this is very simple:

\[
P(C = 1 | B = 1) = \frac{P(C = 1, B = 1)}{P(B = 1)} = \frac{P(B = 1)}{P(B = 1)} = 1
\]

(23)

For the parallel system example,

\[
P(C = 1 | B = 1) = \frac{P(C = 1, B = 1)}{P(B = 1)} = \frac{P(C = 1 | A = 0, B = 1) \times P(A = 0, B = 1)}{P(B = 1)} + \frac{P(C = 1 | A = 1, B = 1)P(A = 1, B = 1)}{P(B = 1)}
\]

\[
= \frac{P\left(R_A \leq \frac{W}{2} \cap R_A \geq \frac{W}{2} \cap R_B \leq \frac{W}{2}\right) + P\left(R_A \leq \frac{W}{2} \cap R_B \leq \frac{W}{2}\right)}{P\left(R_B \leq \frac{W}{2}\right)}
\]

(24)

The joint probabilities in Eqs. (19)–(24) may be obtained by analytical methods of system reliability or Monte Carlo simulation.

3.5. Numerical example

Considering the 2-bar parallel system, it is assumed that \( R_A \), \( R_B \), and \( W \) are independent and normal, with the mean values and standard deviations as shown in Table 3.

Using Monte Carlo simulation, the component failure probabilities and the system failure probability are obtained as

\[
P(A = 1) = P\left(R_A \leq \frac{W}{2}\right) = 0.004957
\]

\[
P(B = 1) = P\left(R_B \leq \frac{W}{2}\right) = 0.001426
\]

Table 3

<table>
<thead>
<tr>
<th></th>
<th>( R_A )</th>
<th>( R_B )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean value (kN)</td>
<td>10.0</td>
<td>12.0</td>
<td>8.0</td>
</tr>
<tr>
<td>Standard deviation (kN)</td>
<td>2.0</td>
<td>2.4</td>
<td>2.4</td>
</tr>
</tbody>
</table>
Monte Carlo simulation yields the same result using the traditional reliability method which represents failure probability as the union of the two possible failure paths:

\[
P(C = 1) = P\left( R_A < \frac{W}{2} \cap R_B < \frac{W}{2} \right) \cup P\left( R_B < \frac{W}{2} \cap R_A < \frac{W}{2} \right) = 0.000061 + 0.0011 + 0.00238 = 0.00354
\]

After a system failure is observed, the component failure probabilities can be updated as

\[
P(A = 1|C = 1) = \frac{P(C = 1|A = 1, B = 1) \times P(A = 1, B = 1)}{P(C = 1)} + \frac{P(C = 1|A = 1, B = 0) \times P(A = 1, B = 0)}{P(C = 1)} = \frac{0.000061 + 0.00238}{0.00354} = 0.69
\]

\[
P(B = 1|C = 1) = \frac{P(C = 1|A = 1, B = 1) \times P(A = 1, B = 1)}{P(C = 1)} + \frac{P(C = 1|A = 0, B = 1) \times P(A = 0, B = 1)}{P(C = 1)} = \frac{0.000061 + 0.0011}{0.00354} = 0.328
\]

The updated probabilistic density functions of the three input random variables \( R_A, R_B, \) and \( W \) using Eq. (22) are shown in Fig. 7.

4. Efficiency of reliability analysis using a Bayesian network

In the previous section, the modified Bayesian network formulation is illustrated through a simple 2-bar system with two failure sequences. For a more complicated structure, the number of combinations of the states (failure or survival) of all components and the number of failure sequences will increase rapidly. This will result in a large computational effort. Consider a simple 3-bar system as shown in Fig. 8(a) for illustration. The states of the three bars are denoted using
A, B, C, and system failure is defined to occur when all the three bars fail, denoted by \( D = 1 \). A Bayesian network similar to Fig. 6 may be constructed. However, in this case, the conditional probability table \( P(D = 1 | A, B, C) \) will have \( 2^3 \) elements. We can easily have
where $C_{(AB)} = 1$, $B_{(AC)} = 1$, $A_{(BC)} = 1$ are the failure events of the third bar after the other two bars have already failed. Their probabilities can be obtained through the updated limit states after the failure of the other two bars. Then we look at the other three conditional probabilities: 

$P(D = 1|A = 0, B = 0, C = 1) = 0$

$P(D = 1|A = 0, B = 1, C = 0) = P(C_{(AB)} = 1|A = 1, B = 1, C = 0)$

$P(D = 1|A = 1, B = 0, C = 1) = P(B_{(AC)} = 1|A = 1, B = 0, C = 1)$

$P(D = 1|A = 0, B = 1, C = 1) = P(A_{(BC)} = 1|A = 0, B = 1, C = 1)$

In the traditional event tree method, the system failure probability is calculated through the union of all the failure sequences. In the Bayesian network analysis, it is seen that partial event tree analysis is also needed for the conditional probability calculation. Therefore, the Bayesian network approach has more computational effort than the event tree approach. However, this extra effort provides a return by enabling backward propagation, that is, reliability re-assessment, when information about the system or any of the nodes becomes available.

With the increase in the number of components, the conditional probability table will enlarge geometrically and many of the terms will involve numerous sub-system failure analyses, which makes the computational implementation quite cumbersome. This restricts the application of a complete Bayesian network to the system reliability analysis of practical structures.

This problem exists even in traditional reliability analysis methods, since numerous potential failure sequences exist in a large-scale structure. In reality, some have high probabilities of occurrence and others may have relatively low probabilities of occurrence. In such a case, the
reliability of the structural system is evaluated by selecting the probabilistically dominant failure paths. Several methods, such as the branch and bound [5] or truncated enumeration [6] may be used for this selection. A failure sequence with a low failure probability is truncated before proceeding to system failure, to avoid further enumeration. In this way, the insignificant failure sequences are discarded and the calculation of system failure probability is simplified.

This paper uses the branch and bound concept for the construction of the conditional probability table in the Bayesian network. The effects of the events with relatively very small probabilities are ignored with their probabilities set to be zero. This approach is illustrated through a frame structure shown in Fig. 10. Structural failure is defined as the formation of a collapse mechanism for which the bending moment is dominant. It is assumed that the loads $P_1$ and $P_2$ are uncorrelated normal-distributed random variables and so are the bending moment capacities of the three frame members. For illustrative purposes, all the numerical data are taken the same as in [5] and are shown in Table 4.

The Bayesian network corresponding to the frame is constructed as in Fig. 11, in which $S$ denotes the state of the system. There are two states for the components and the system: 1 representing failure and 0 representing survival. The failure domains $Fi$ of the nodes 1, 2, 3, 4, 6, 7 and 8 are obtained as
The conditional probability table \( P(S=1|F_1, F_2, \ldots, F_8) \) will have \( 2^7 = 128 \) elements which need to be obtained through system reliability analysis separately. To make the computational implementation feasible, the branch and bound method is used to select the significant failure sequences and discard unimportant failure sequences and combinations of states of \( (F_1 \ldots F_8) \). Five complete failure paths are selected, as shown in Fig. 12.

From this, it is seen that the failure sequences originating from the failure of 1, 2, 3, and 6, are all discarded due to their relatively low probability of occurrence. This means that the probabilities \( P(F_1 = 1) \), \( P(F_2 = 1) \), \( P(F_3 = 1) \), and \( P(F_6 = 1) \) are all very low and are ignored. Since the marginal probability \( P(F_1 = 1) \) is very low, the joint probability of \( F_1 = 1 \) and any combination of states of the other six nodes which is represented using \( P(F_1 = 1, F_2, F_3, F_4, F_6, F_7, F_8) \) in this paper, is even lower and approximated as zero in the conditional probability table. Similarly, \( P(F_1, F_2 = 1, F_3, F_4, F_6, F_7, F_8) = 0 \), \( P(F_1, F_2, F_3 = 1, F_4, F_6, F_7, F_8) = 0 \), and \( P(F_1, F_2, F_3, F_4, F_6 = 1, F_7, F_8) = 0 \).

### Table 4

Data for the frame structure

<table>
<thead>
<tr>
<th>Plastic hinge location</th>
<th>Cross section area (m²)</th>
<th>Moment of inertia (m⁴)</th>
<th>Mean value of strength ( R ) (kNm)</th>
<th>COV of strength ( R ) (kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2</td>
<td>4.00 \times 10⁻³</td>
<td>3.58 \times 10⁻⁵</td>
<td>75.0</td>
<td>0.05</td>
</tr>
<tr>
<td>3,4</td>
<td>4.00 \times 10⁻³</td>
<td>4.77 \times 10⁻⁵</td>
<td>101.0</td>
<td>0.05</td>
</tr>
<tr>
<td>5,6</td>
<td>4.00 \times 10⁻³</td>
<td>4.77 \times 10⁻⁵</td>
<td>101.0</td>
<td>0.05</td>
</tr>
<tr>
<td>7,8</td>
<td>4.00 \times 10⁻³</td>
<td>3.58 \times 10⁻⁵</td>
<td>75.0</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Young’s modulus: 210 Gpa
Mean value of yield stress \( \bar{\sigma}_y = 276 \) MPa
\( \bar{\tau}_c = 20 \) kN, \( \bar{\tau}_c = 40 \) kN, \( \text{COV}_{R_i} = 0.05/0.3 \), \( \text{COV}_{R_p} = 0.05/0.3 \)
Finally, there are 6 joint probabilities to be considered. Furthermore, \( P(S = 1 \mid F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 0, F_8 = 0) = 0 \), therefore the probability \( P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 0, F_8 = 0) \) need not be calculated. Thus, using the total probability theorem, system reliability estimation is reduced to the sum of five products (five joint probabilities multiplied by the corresponding conditional system failure probability) shown below:

\[
P(S = 1) = P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 0, F_8 = 0) \\
+ P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 1, F_8 = 0) \\
+ P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 0) \\
+ P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 1) \\
+ P(F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 1, F_8 = 1)
\]

Each of these products can be reduced to a joint probability:

\[
P(S = 1) = P(S = 1, F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 0, F_8 = 0) \\
+ P(S = 1, F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 1, F_8 = 0) \\
+ P(S = 1, F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 0) \\
+ P(S = 1, F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 1) \\
+ P(S = 1, F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 1, F_8 = 1)
\]

Finally, these joint probabilities are simply the intersection of an event (or set of events) with the union of paths initiated by that event (or set of events).

\[
P(S = 1) = P\left\{ F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 0, F_8 = 0, (M_{7(4)} < 0, M_{8(4)} < 0, M_{2(74)} < 0) \cup (M_{7(4)} < 0, M_{8(4)} < 0) \right\} \\
+ P\left\{ F_1 \ldots F_6 = 0, F_7 = 1, F_8 = 0, (M_{4(7)} < 0, M_{8(47)} < 0, M_{2(847)} < 0)(M_{4(7)} < 0, M_{2(47)} < 0) \right\} \\
+ P\left\{ F_1 \ldots F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 0, (M_{2(47)} < 0)(M_{8(47)} < 0, M_{2(847)} < 0) \right\} \\
+ P\left\{ F_1 \ldots F_3 = 0, F_4 = 1, F_6 = 0, F_7 = 1, F_8 = 1, (M_{2(847)} < 0) \right\} \\
+ P\left\{ F_1 \ldots F_3 = 0, F_4 = 0, F_6 = 0, F_7 = 1, F_8 = 1, (M_{4(78)} < 0, M_{2(478)} < 0) \right\}
\]

where, for example, \( M_{7(4)} \) is the updated performance function for a plastic hinge to form at location 7 after one has already formed at location 4. The notation is similar for other performance functions.
Using Monte Carlo simulation, system failure is computed as

\[ P(S = 1) = 0.0003 + 0.0054 + 0.009 + 0 = 0.0066 \]

The same system failure can be calculated with the traditional method of taking the union of the five significant failure paths:

\[
P(S = 1) = P\{ (M_7 < 0, M_{4(7)} < 0, M_{2(847)} < 0) \} \
\cup
M_7 < 0, M_{4(87)} < 0, M_{2(487)} < 0 \} \cup (M_4 < 0, M_{7(4)} < 0, M_{8(74)} < 0, M_{2(874)} < 0) \\
\cup (M_4 < 0, M_{7(4)} < 0, M_{2(74)} < 0) \} = 0.0066
\]

During forward propagation to calculate system failure, the proposed method does not provide any result different from the traditional system reliability method. In fact, it requires extra calculations to compute the probabilities of several subsystems (for example, the three subsystems a, b, and c in Fig. 12). The advantage of this extra computation is that it facilitates backward propagation for updating the conditional probabilities after system performance information is obtained. In the forward propagation, the estimates of overall system failure probability, branch probabilities, and correlation effects will be same as that obtained using the traditional system reliability method.

As can be seen, the use of the branch and bound approach makes the proposed Bayesian network method feasible for reliability reassessment and updating. The accuracy of the approximation depends on the criterion used to discard a failure sequence in the branch-and-bound approach.

Given new information on system performance (failure or survival), the probability of plastic hinge formation at a specific location can be updated. For example, if system collapse is observed, we obtain

\[
P(F4 = 1|S = 1) = \frac{P(F4 = 1, S = 1)}{P(S = 1)} = \frac{0.66}{0.66} = 1
\]

\[
P(F7 = 1|S = 1) = \frac{P(F7 = 1, S = 1)}{P(S = 1)} = \frac{0.0064}{0.0066} = 0.970
\]

Note that in the above forward and backward propagation, all the terms corresponding to the joint probabilities, \( P(F1 = 1, F2, F3, F4, F6, F7, F8) \), \( P(F1, F2 = 1, F3, F4, F6, F7, F8) \), \( P(F1, F2, F3 = 1, F4, F6, F7, F8) \), and \( P(F1, F2, F3, F4, F6 = 1, F7, F8) \), are omitted due to their relatively low values. This approximation will result in the updated failure probabilities \( P(F1 = 1|S = 1) \), \( P(F2 = 1|S = 1) \), \( P(F3 = 1|S = 1) \) and \( P(F6 = 1|S = 1) \) all equal to zero. Since 4 and 7 are the two weakest locations where plastic hinge most probably forms, their updated failure probabilities are of most concern, compared to the other locations. However, in the case that the failure probabilities—original evaluation and/or posterior updating—at other locations also need to be considered, the branch-and-bound method may be modified by previewing all the potential plastic hinges after the first branching operation, and bounding only in the subsequent steps.
5. Conclusion

A new methodology has been developed in this paper for the application of the Bayesian network concept to structural system reliability reassessment. Multiple failure sequences and correlations between component failures are considered and incorporated in the Bayesian network construction. The proposed approach to include these two additional features is validated by comparing it with the results of traditional reliability analysis. Both forward and backward propagation formulae are developed. In order to extend the method to the application of more complicated structures, the Bayesian network is combined with the branch-and-bound method to consider only the significant failure sequences. A frame structure with multiple potential locations of plastic hinges and multiple failure sequences is analyzed to illustrate the proposed approach. The numerical result shows that the combination of Bayesian network and branch and bound approach considerably improves the efficiency of the method and makes it applicable to the reliability reassessment of large structures, when new data on structural performance becomes available.

The proposed reliability reassessment approach applies the Bayesian network to both forward and backward uncertainty propagation between component and system performance, and solves the problem of information transformation from system level to component level. Since most real structures involve multiple components and multiple failure sequences, the proposed modifications in the Bayesian network facilitate a comprehensive framework for system reliability reassessment.

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